Lecture 23

11.3/11.4 - The Integral and Comparison Tests

We can use integrals to study the convergence of series by comparing the two. The comparison test for series looks very much like the one for integrals. The integral test applies very generally, but we will only make limited use of it. Here is the general statement:

Integral Test

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges. I.e.:

ⓐ $\int_1^{\infty} f(x)\,dx$ convergent $\Rightarrow \sum_{n=1}^{\infty} a_n$ convergent

ⓑ $\int_1^{\infty} f(x)\,dx$ divergent $\Rightarrow \sum_{n=1}^{\infty} a_n$ divergent

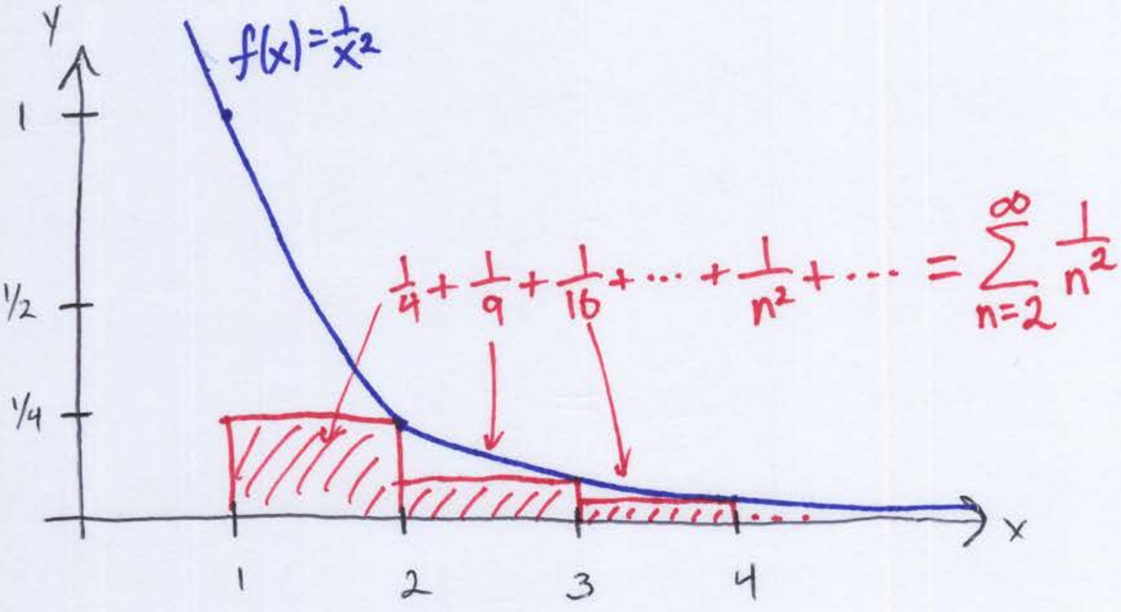
Let's look at the following two examples to get an idea of why this theorem works:

Recall: $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$
- converges if $p>1$
- diverges if $p\le 1$

Ex (a): Does $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converge?

We compare $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ to $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$.

Let's approximate $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ using right endpoints and step size $\Delta x = 1$.



Notice that the right endpoints give a lower approximation to the integral. Thus:

$$\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx$$

which implies:

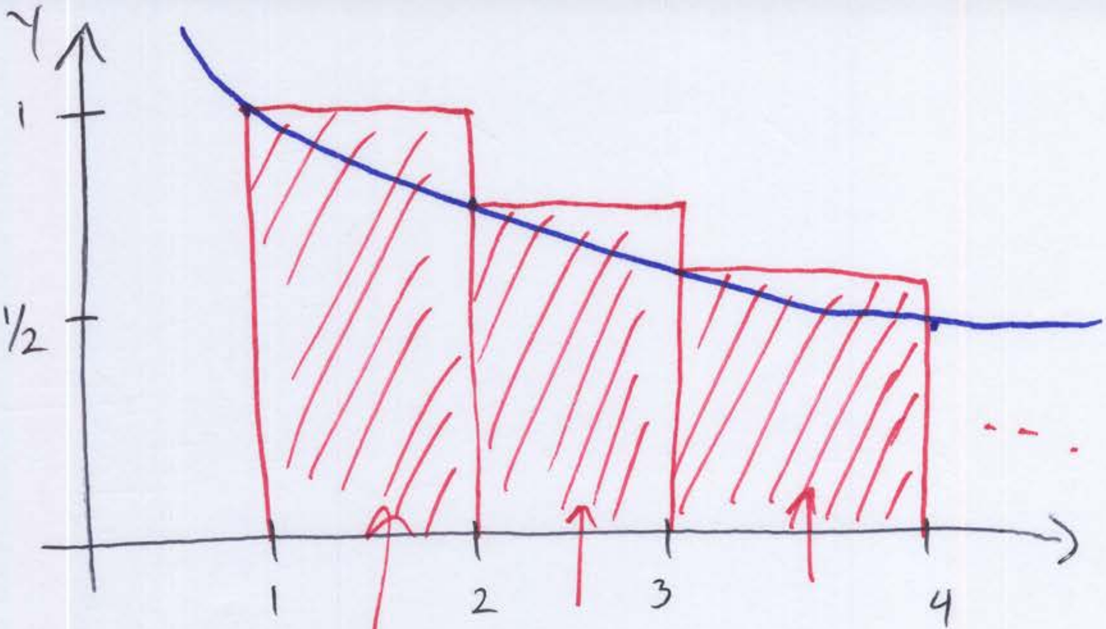
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

So, since we know that $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges (i.e., is finite), this implies that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is as well. Thus $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Ex (b): Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge?

This time we compare to $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$, which we know diverges. We use the same idea as last time, but this time we use left endpoints so we get an overestimate of the integral:

Taking $\Delta x = 1$ again:



$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

So, we get $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$

But $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ diverges (to infinity), thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ does as well.

This theorem, together with our knowledge of $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ gives us the <u>p-series test</u>:

> The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Example: Which of the following converge? Diverge?

ⓐ $\sum_{n=1}^{\infty} \frac{1}{n}$ ⓑ $\sum_{n=1}^{\infty} n^{-5/2}$ ⓒ $\sum_{n=500}^{\infty} \sqrt{n}$

ⓓ $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

Sol: ⓐ Here $p=1$, so the series diverges by the p-series test.

ⓑ $\sum_{n=1}^{\infty} n^{-5/2} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$. Here $p=\frac{5}{2}$, so the series converges by the p-series test.

ⓒ $\sum_{n=500}^{\infty} \sqrt{n} = \sum_{n=500}^{\infty} \frac{1}{n^{-1/2}}$. $p = \frac{-1}{2} \Rightarrow$ series diverges

ⓓ $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}$.

$p=3 \Rightarrow$ series converges.

Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with $a_n \geq 0$ & $b_n \geq 0$.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $b_n \leq a_n$ for all n, then $\sum a_n$ is also divergent.

proof: Let $s_n = \sum_{j=1}^{n} a_j$ & $t_n = \sum_{j=1}^{n} b_j$.

(i) Since $\sum b_n$ converges, we have $\sum_{j=1}^{\infty} b_j = t = \lim_{n\to\infty} t_n$.

Since $a_n, b_n \geq 0$ for all n, the sequences $\{s_n\}$ & $\{t_n\}$ are increasing. Since $a_n \leq b_n$ for all n, $s_n \leq t_n$ for all n. Since t_n is increasing $s_n \leq t_n \leq t$ for all n. Thus $0 \leq s_n \leq t$ for all n, hence $\{s_n\}$ is a bounded monotonic sequence, and thus converges. Therefore, $\sum a_n$ converges.

(ii) Similar ideas. Try at home!

$\square$

Ex: Do the series

ⓐ $\sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3}$ ⓑ $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ ⓒ $\sum_{n=1}^{\infty} \frac{6}{3n^2+7n-2}$

ⓓ $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ ⓔ $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$

converge?

Sol: ⓐ Since $2^{-1/n} = \frac{1}{2^{1/n}} \leq 1$ for all n, we have $\frac{2^{-1/n}}{n^3} \leq \frac{1}{n^3}$ for all n. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (by p-series test), so does $\sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3}$.

ⓑ $\frac{1}{n^2+1} \leq \frac{1}{n^2}$ (b/c $n^2+1 \geq n^2$) for all n.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ by p-series test.

So $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

ⓒ $\dfrac{6}{3n^2+7n-2} \leq \dfrac{6}{3n^2} = \dfrac{2}{n^2}$ (b/c $7n-2 \geq 0$ for $n \geq 1$
thus $3n^2+7n-2 \geq 3n^2$)

So, $\displaystyle\sum_{n=1}^{\infty} \frac{6}{3n^2+7n-2} \leq \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty$ by p-series test.

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{6}{3n^2+7n-2}$ converges.

ⓓ $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ (In class, we went out to $n=4$, but $n=3$ is
fine since $\ln 3 > 1$.)

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n} = \frac{\ln(1)}{1} + \frac{\ln(2)}{2} + \sum_{n=3}^{\infty} \frac{\ln(n)}{n} = \frac{\ln(2)}{2} + \sum_{n=3}^{\infty} \frac{\ln(n)}{n}$$

Since $\ln(n) \geq 1$ for $n \geq 3$, $\dfrac{\ln(n)}{n} \geq \dfrac{1}{n}$ for $n \geq 3$,

we have:

$\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n} \geq \frac{\ln(2)}{2} + \sum_{n=3}^{\infty} \frac{1}{n}$ diverges by p-series test.

So $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges.

ⓔ $3+10^n \geq 10^n$ for all n, so $\dfrac{9^n}{3+10^n} \leq \dfrac{9^n}{10^n}$ for all n.

So $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n} \leq \sum_{n=1}^{\infty} \frac{9^n}{10^n} = \sum_{n=1}^{\infty} \frac{9}{10}\left(\frac{9}{10}\right)^{n-1} < \infty$ convergent geometric series.

So $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ converges.